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# Computing with Differential-difference Operators

CHARLES F. DUNKL

*Department of Mathematics, University of Virginia, Charlottesville, VA 22903, U.S.A.*

Computer algebra can be used to prove identities in the algebra of operators on polynomials which is generated by multiplication by coordinate functions, and the group translations and Dunkl operators associated with a reflection group. This technique is illustrated by a conceptual proof of a complete orthogonal decomposition of the harmonic polynomials associated with the abelian reflection groups.

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## 1. Analysis on Root Systems

Polynomial algebra can generally be done very efficiently by modern computer algebra systems. Computing with rational functions, however, slows down the process, especially if there are many variables and several different denominators. Much interesting analysis on root systems depends on dividing by linear functions whose zero-sets are the mirrors of the corresponding reflection group. In this paper we present a computational scheme which can be used to discover and prove operator identities. The method is illustrated by exhibiting a set of commuting self-adjoint operators for the spherical harmonic polynomials associated to a $\mathbb{Z}_2^N$-type weight function. The eigenfunctions are expressed as products of Jacobi polynomials.

We first define the differential-difference operators (called “Dunkl operators” in the literature). Here is the background: for a non-zero vector $v \in \mathbb{R}^N$ the associated reflection $\sigma_v$ is defined by $x\sigma_v := x - 2(\langle x, v\rangle/|v|^2)v$ for $x \in \mathbb{R}^N$, an orthogonal transformation; where $\langle x, y\rangle := \sum_{i=1}^N x_i y_i$ is the inner product and $|v|^2 = \langle v, v\rangle$. A root system is a finite set $R$ of non-zero vectors which is closed under each $\sigma_v, v \in R$; here it is further assumed that the root system is reduced, that is, if $v, v' \in R$ and $v' = cv$ for some $c \in \mathbb{R}$ then $c = \pm 1$. A set $R_+$ of positive roots is defined as follows: fix $u \in \mathbb{R}^N$ such that $\langle u, v\rangle \neq 0$ for each $v \in R$, then $R_+ := \{v \in R : \langle u, v\rangle > 0\}$. The finite reflection (or Coxeter) group associated with the root system is a finite subgroup of the orthogonal group and is denoted by $W = W(R)$, the group generated by $\{\sigma_v : v \in R_+\}$. For analysis purposes define a multiplicity function $\{k_v : v \in R\}$ where the values are either formal parameters or nonnegative real numbers, and the values are constant on conjugacy classes, that is, if $u, v \in R$ and $v = uw$ for some $w \in W$ then $k_u = k_v$. For indecomposable Coxeter groups the number of conjugacy classes is one or two. In the subsequent definitions and theorems the root system and multiplicity function will be assumed to satisfy the stated properties.

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Definition 1.1. For $i = 1, \ldots, N$ the first-order differential-difference operator is defined by

$$T_i f(x) := \frac{\partial f(x)}{\partial x_i} + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} v_i,$$

where $f(x)$ is a polynomial or an entire function. When $f$ is a homogeneous polynomial then $T_i f$ is also homogeneous of degree one less.

The author constructed these operators and showed that they commute pairwise in Dunkl (1989). The commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero–Sutherland–Moser models, which deal with systems of identical particles in a one-dimensional space (for example, see Hikami, 1996; Kakei, 1996; Lapointe and Vinet, 1996). Here we are concerned with the algebra $\mathbb{A}$ of operators on polynomials generated by $\{T_i : i = 1, \ldots, N\}$, $\{x_i, i = 1, \ldots, N\}$ (multiplication), $\{\sigma_v : v \in R_+\}$. This is an analogue of the Weyl algebra which is generated by $\{\frac{\partial}{\partial x_i}, x_i : i = 1, \ldots, N\}$. We desire to use a computer algebra system to experiment in an attempt to discover commuting operators, or identities in this algebra. The straightforward approach would be to apply $A_1, A_2 \in \mathbb{A}$ to a general (undefined) function $f(x)$ and check whether $A_1 f(x) = A_2 f(x)$. As stated before, this usually results in a linear combination of various partial derivatives of $\{f(xw) : w \in W\}$ with rational function coefficients. This can lead to very large expansions. Typically we might investigate commutation relationships among second-order operators, which could involve fourth-order terms. We proceed to describe our method for checking identities in the algebra $\mathbb{A}$; it depends on an analogue of the exponential function. We need a product rule.

Proposition 1.2. *For appropriate functions $f, g$ on $\mathbb{R}^N$ and $i = 1, \ldots, N$,*

$$T_i(fg)(x) = f(x) T_i g(x) + g(x) \frac{\partial f(x)}{\partial x_i} + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} g(x\sigma_v) v_i.$$

Proof. This statement is proven by direct verification.□

The idea is to mimic the behavior of the function $x \mapsto \exp(\langle x, y\rangle)$ with differentiation replaced by the operators $\{T_i\}$. This is possible with the function $K(x, y)$ constructed in Dunkl (1991). Let $\mathcal{P}_n$ denote the space of homogeneous polynomials on $\mathbb{R}^N$ of degree $n$; there exists a unique linear operator $V : \mathcal{P}_n \to \mathcal{P}_n$ for each $n \in \mathbb{Z}_+$ such that $T_i V = V \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, N$ and $V1 = 1$. This is called the intertwining operator. The kernel $K(x, y)$ is defined by $K_n(x, y) := V_x(\langle x, y\rangle^n / n!)$ and $K(x, y) := \sum_{n=0}^{\infty} K_n(x, y)$, where $V_x$ acts on the variable $x$, for $x, y \in \mathbb{R}^N$. The kernel has the following properties (for $k_v \geq 0, n \in \mathbb{Z}_+$, $x, y \in \mathbb{R}^N$, $w \in W$) Dunkl (1991, Proposition 3.2 and Corollary 3.3)

1. $|K_n(x, y)| \leq (|x||y|)^n / n!$,
2. $K(x, y) = K(y, x)$,
3. $K(xw, yw) = K(x, y)$,
4. $T_{x,i} K(x, y) = y_i K(x, y)$ (acting on the variable $x$),
5. $K_n(x, T_y) p(y) = p(x)$ for each $p \in \mathcal{P}_n$.

The reproducing property in the last item means that in $K_n(x, y)$ each $y_i$ is replaced by $T_{y,i}$, acting on the variable $y$. This is the basis for the following which is fundamental for this paper.

THEOREM 1.3. *Suppose $A \in \mathbb{A}$, then $Ap = 0$ for every polynomial $p$ iff $A_x K(x, y) = 0$ for each $y \in \mathbb{R}^N$ (the subscript on $A$ indicates the action variable).*

PROOF. For any fixed $y \in \mathbb{R}^N$ the function $x \mapsto K_n(x, y)$ is a polynomial. The bound (1) shows that $A_x K(x, y) = \sum_{n=0}^{\infty} A_x K_n(x, y)$ because the series is uniformly and absolutely convergent for $x \in \mathbb{R}^N$. Thus, $A_x p(x) = 0$ for each polynomial $p$ implies $A_x K(x, y) = 0$.

Now suppose $A_x K(x, y)$ is fully expanded using Proposition 1.2. The result is a linear combination of $\{K(xw, y) : w \in W\}$ with coefficients which are polynomials in $x, y, k_v$. For example, $T_{x,i} K(xw, y) = T_{x,i} K(x, yw^{-1}) = (yw^{-1})_i K(xw, y)$, using property (3). The effect of the operator $A$ on a polynomial $p(y)$ can be found by replacing each $y_i$ in $A_x K(x, y)$ by $T_{y,i}$, applying this to $p(y)$ and setting $y = 0$. This has the effect of using the reproducing property (5) on each homogeneous component of $p(y)$. This shows the other half of the theorem.□

The computation of $A_x K(x, y)$ can be done in Maple$^{TM}$ V, Release 4, by introducing an undefined function with argument $\langle x, y\rangle$ and differentiating the argument with respect to $x_i$ to obtain the multiplier, and substituting $x\sigma_v$ for $x$ as called for in Proposition 1.2. For example, define f := K(x[1] * y[1] + x[2] * y[2]); then $T_1 K$ is computed by diff(op(1, f), x[1]) * f, and a typical reflection calculation is subs({x[1]=x[2],x[2]=x[1]},f). The following Maple procedures illustrate how one can separate a product of polynomials with $K$ and implement the product rule. We choose the root system $A_{N-1}$ as example; this generates the symmetric group on $N$ objects; there is one conjugacy class of reflections (usually called transpositions) and one parameter $k$ for the multiplicity function.

Transpositions are denoted by $(ij)$, defined by $x(ij) := (x_1, \ldots, \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots, x_N)$ for $x \in \mathbb{R}^N$. In this case the formula for $T_i$ is

$$T_i p(x) = \frac{\partial p(x)}{\partial x_i} + k \sum_{j \neq i} \frac{p(x) - p(x(ij))}{x_i - x_j}.$$

Of course the computations must be done for specific values of $N$. The procedures defined here can be used to show the known Dunkl (1998, Lemma 2.5) commutation relation $[T_1 x_1, T_2 x_2 - k(12)] = 0$ for the symmetric group on $\mathbb{R}^4$ (or some other specific dimension). These procedures are intended to be applied to sums of products of monomials in $x, y, k_v$ times a term of the form $K(\langle xw, y\rangle)$ for some $w \in W$. The first procedure separates out the monomial, and the second one implements $T_i(f)$ for the symmetric group on $\mathbb{R}^n$. The letter $a$ denotes Weyl groups of type A.

```
getpol:=proc(g) local g1,p1,p2,i; p1 := 1; p2 := 1;
if type(g,`*`) then for i to nops(g) do
g1 := op(i,g) ; if type(g1, polynom) then p1 := p1*g1 else p2 := p2*g1 fi od;
else p2 := g fi; [p1,p2] end;
```

```
ta := proc(f,i,n) local fx, i1, j1, tm, tm1, tm2, sm, sm1; global k,x,getpol;
fx := expand(f+1); sm := 0;
for i1 to nops(fx) do
tm := getpol(op(i1,fx)) ; tm1 := tm[1]; tm2 := tm[ 2];
sm1 := diff(tm1, x[i])* tm2 + tm1*diff(op(1,tm2), x[i ])*tm2;
for j1 to n do
if j1 ≠ i then sm1 := sm1 + k*quo(tm1- subs({x[i ]=x[j1], x[j1]=x[i]} , tm1), x[i]-
x[j1], x[ i]) * subs({x[i]=x[ j1], x[j1]=x[i]}, tm2) fi od;
sm := sm + sm1 od; sm end;
```

The purpose of the line *fx*:=expand*(f+1)* is a quick way of forcing the input to be a sum; the function 1 is later discarded. The innermost loop ($j1$) can be made more efficient by use of the collect facility, and doing the division by a subroutine which implements the formula (with $a, b \in \mathbb{Z}_+$)

$$\frac{x_i^a x_j^b - x_i^b x_j^a}{x_i - x_j} = sign(a-b) \sum_{s=\min(0,a-b)}^{\max(-1,a-b-1)} x_i^{a-1-s} x_j^{b+s}.$$

## 2. Spherical Harmonics

For a root system and multiplicity function as described above, let $h(x) = \prod_{v \in R_+} |\langle x, v\rangle|^{k_v}$, a positively homogeneous $W$-invariant function and consider the Hilbert space $\mathcal{H} := L^2(S, h^2 dm)$, where $S$ is the unit sphere in $\mathbb{R}^N$ and $dm$ is the rotation-invariant measure on $S$. It was shown in Dunkl (1988) that there is an orthogonal decomposition of this space as the direct sum $\mathcal{H} = \sum_{n=0}^{\infty} \oplus \{p : p \in \mathcal{P}_n, \Delta_k p = 0\}$, where the Laplacian-type operator is $\Delta_k := \sum_{i=1}^{N} T_i^2$. Further orthogonal decomposition could be done with a complete set of commuting self-adjoint operators on $\mathcal{H}$. At this writing these are not known for root systems of rank $\geq 3$; the main goal of this paper is to provide tools for further research. The situation for $W = (\mathbb{Z}_2)^N$, the direct product, has been thoroughly studied and we will exhibit the complete orthogonal decomposition. First we show how the classical Jacobi polynomials arise in the problem of constructing harmonic polynomials for decomposable reflection groups (this means that $W = W_1 \times W_2$, each factor is a reflection group and the two factors act on perpendicular subspaces). There is a product rule for $\Delta_k$ for the general case.

Proposition 2.1. *For polynomials $f, g$ on $\mathbb{R}^N$,*

$$\Delta_k(fg) = f\Delta_k g + g\Delta_k f + 2\langle \nabla f, \nabla g\rangle + \sum_{v \in R_+} k_v \frac{(f - \sigma_v f)(g - \sigma_v g)}{\langle x, v\rangle^2} |v|^2.$$

Proof. The formula is an application of the expression for $\Delta_k$ derived in Dunkl (1989). The function $\sigma_v f(x) := f(x\sigma_v)$ and $\nabla$ denotes the gradient.□

Suppose that $W_1, W_2$ act on $(x_1, \ldots, x_m)$ and $(x_{m+1}, \ldots, x_N)$ respectively; that their respective sets of positive roots are $R_{1,+}$ and $R_{2,+}$; then let $t_1 := \sum_{i=1}^{m} x_i^2, t_2 := \sum_{i=m+1}^{N} x_i^2$ and $\lambda_1 := \sum_{v \in R_{1,+}} k_v$, $\lambda_2 := \sum_{v \in R_{2,+}} k_v$.

PROPOSITION 2.2. *Suppose* $p_1(x_1, \ldots, x_m)$ *and* $p_2(x_{m+1}, \ldots, x_N)$ *are harmonic, that is,* $\sum_{i=1}^{m} T_i^2 p_1 = 0$ *and* $\sum_{i=m+1}^{N} T_i^2 p_2 = 0$ *then* $\Delta_k(f(t_1, t_2)p_1p_2) = 0$ *iff* $f$ *satisfies the differential equation*
$(t_1(\frac{\partial}{\partial t_1})^2 + t_2(\frac{\partial}{\partial t_2})^2 + (\lambda_1 + \deg(p_1) + \frac{m}{2})\frac{\partial}{\partial t_1} + (\lambda_2 + \deg(p_2) + \frac{N-m}{2})\frac{\partial}{\partial t_2})f(t_1, t_2) = 0.$

PROOF. Because $p_1$ and $p_2$ depend on disjoint sets of variables the product rule for $\Delta_k$ can be applied separately to $fp_1$ and $fp_2$. Thus $\langle \nabla f, \nabla p_1 \rangle = 2\sum_{i=1}^{m} \frac{\partial f}{\partial t_1} x_i \frac{\partial p_1}{\partial x_i} = 2\frac{\partial f}{\partial t_1} \deg(p_1)p_1$ (and similarly for $p_2$), $\sum_{i=1}^{m} T_i^2 f(t_1, t_2) = 4t_1 \frac{\partial^2 f}{\partial t_1^2} + (2m + 4\lambda_1)\frac{\partial f}{\partial t_1}$, using the invariance of $f$ (and similarly for $i = m+1, \ldots, N$).□

COROLLARY 2.3. *Under the hypotheses of the proposition,* $\Delta_k(f(t_1, t_2)p_1p_2) = 0$ *for* $f(t_1, t_2) = (t_1+t_2)^n P_n^{(\alpha,\beta)}\left(\frac{t_1-t_2}{t_1+t_2}\right)$ *for* $n = 0, 1, 2, 3, \ldots$*, where* $\alpha := \frac{N-m}{2} + \lambda_2 + \deg(p_2) - 1$ *and* $\beta := \frac{m}{2} + \lambda_1 + \deg(p_1) - 1$ *and* $P_n^{(\alpha,\beta)}$ *denotes the Jacobi polynomial.*

PROOF. Write $f$ as a homogeneous polynomial of degree $n$ with undetermined coefficients $a_i$, that is, $f(t_1, t_2) = \sum_{i=0}^{n} a_i t_1^{n-i} t_2^i$ ; the differential equation implies $a_{i+1} = -\frac{(n-i)(n-i+\beta)}{(i+1)(i+\alpha+1)} a_i$ for $i = 0, \ldots, n-1$; with the solution $a_i = (-1)^i \frac{(-n)_i(-n-\beta)_i}{(\alpha+1)_i \cdot i!} a_0$ (in Pochhammer symbol notation). A standard transformation of hypergeometric series establishes the Jacobi polynomial formula.□

Given bases of harmonic homogeneous polynomials for $W_1$ and $W_2$ this construction produces a basis for $W_1 \times W_2$; we show this with a dimension argument. Koornwinder (1973, Theorem 4.2) stated and proved the Corollary for the standard case (all $k_v = 0$) and proved the following Lemma by a generating-function argument. Let $d(N, n)$ denote the cardinality of $\{\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N : \sum_{i=1}^{N} \alpha_i = n\}$; this is of course also $\dim(\mathcal{P}_n)$ and equals $\binom{N+n-1}{n}$. The dimension of $\{p \in \mathcal{P}_n : \Delta_k p = 0\}$ equals $d(N, n) - d(N, n-2)$. Note that the harmonic polynomial in the Corollary is of degree $\deg(p_1) + \deg(p_2) + 2n$.

LEMMA 2.4. $d(N, n) - d(N, n-2) = d(N-1, n) + d(N-1, n-1) = \sum\{(d(m, i) - d(m, i-2))(d(N-m, n-i-2j) - d(N-m, n-i-2j-2)) : i + 2j \leq n\}$*, for fixed* $m = 1, \ldots, N-1$*. The boundary values are* $d(m, 0) = 1$ *for* $m \geq 0$*,* $d(m, n) = 0$ *for* $m < 0$ *and* $n \geq 0$*.*

PROOF. For the first equation suppose $\alpha \in \mathbb{Z}_+^N$ and $\sum_{s=1}^{N} \alpha_s = n$. Then one of the following holds: $\alpha_N \geq 2, \alpha_N = 1, \alpha_N = 0$; this shows that $d(N, n) = d(N, n-2) + d(N-1, n-1) + d(N-1, n)$ (for the first case replace $\alpha_N$ by $\alpha_N - 2$, for the other cases consider $(\alpha_1, \ldots, \alpha_{N-1})$. For the second equation replace the sum by $\sum\{(d(m-1, i) + d(m-1, i-1)) \cdot (d(N-m-1, n-i-2j) + d(N-m-1, n-i-2j-1)) : i + 2j \leq n\}$ and expand the products. For any $(\alpha_1, \ldots, \alpha_{N-1})$ with $\sum_{s=1}^{N-1} \alpha_s = n$ or $n-1$ form the triple $(\sum_{s=1}^{m-1} \alpha_s, \alpha_m, \sum_{s=m+1}^{N-1} \alpha_s)$; these can be split into 4 cases $(i, 2j, n-i-2j)$, $(i, 2j+1, n-i-2j-1)$, $(i-1, 2j, n-i-2j)$, $(i-1, 2j+1, n-i-2j-1)$, the first two with total $n$, the second two with total $n-1$. This proves the equation in a bijective manner.□

One implication is that the number of linearly independent harmonic polynomials in $\mathcal{P}_n$ equals the number of $N$-tuples $\alpha$ with $\alpha_N = 0$ or 1 and $\sum_{i=1}^N \alpha_i = n$.

Henceforth, $W = \mathbb{Z}_2^N$ with the reflections $\sigma_i$ defined by $x\sigma_i := (x_1, \dots, -x_i, \dots, x_N)$; the multiplicity function is described by $(k_1, \dots, k_N)$ and $T_i f(x) = \frac{\partial f(x)}{\partial x_i} + k_i \frac{f(x) - f(x\sigma_i)}{x_i}$ for $i = 1, \dots, N$. The weight function for $\mathcal{H}$ is $\prod_{i=1}^N |x_i|^{2k_i}$. We will construct an orthogonal basis of harmonic homogeneous polynomials labeled by $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_N = 0$ or 1 and then present a set of commuting self-adjoint operators for which these polynomials are simultaneous eigenfunctions.

Definition 2.5. For each $\alpha \in \mathbb{Z}_+^N$ with $\alpha_N = 0$ or 1, there is a harmonic homogeneous polynomial $p(\alpha; x)$, of degree $\sum_{i=1}^N \alpha_i$, constructed inductively as follows: let $\alpha_i = 2a_i + \epsilon_i$, for each $i$, with $a_i \in \mathbb{Z}_+$ and $\epsilon_i = 0$ or 1 (and $a_N = 0$); $p((0, \dots, \epsilon_N); x) = x_N^{\epsilon_N}$, and when $\alpha_1 = \alpha_2 = \dots = \alpha_{N-m-2} = 0$ for some $m$, let $\alpha' := (0, \dots, 0, \alpha_{N-m}, \dots, \alpha_N)$, let $\gamma := \frac{m+1}{2} + \sum_{i=N-m}^N (k_i + \alpha_i) - 1$, $\delta := k_{N-m-1} + \epsilon_{N-m-1} - \frac{1}{2}$, $t_1 := x_{N-m-1}^2$, $t_2 := \sum_{i=N-m}^N x_i^2$, and $s := a_{N-m-1}$ ; then $p(\alpha; x) := (t_1 + t_2)^s P_s^{(\gamma,\delta)}\left(\frac{t_1 - t_2}{t_1 + t_2}\right) \cdot (x_{N-m-1})^{\epsilon_{N-m-1}} p(\alpha'; x)$.

Note that $p(\alpha; x)$ is a linear combination of monomials $x^\beta := \prod_{i=1}^N x_i^{\beta_i}$ such that $\beta_i \equiv \alpha_i \bmod 2$ for each $i$ and $\alpha \succeq \beta$, that is, $\sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i$ for $j = 1, \dots, N$, the dominance order for compositions. In the inductive definition $p(\alpha; x)$ is obtained from $p(\alpha'; x)$ by multiplying by a sum of terms like $x_{N-m-1}^{\alpha_{N-m-1} - 2j} (\sum_{i=N-m}^N x_i^2)^j$, a typical monomial in the product has exponent $(0, \dots, \alpha_{N-m-1} - 2j, \beta'_{N-m}, \dots, \beta'_N)$ where $\beta'$ is obtained from $\beta = (0, \dots, 0, \beta_{N-m}, \dots, \beta_N)$ by adding 2 to the components $j$ times, and inductively $(0, \dots, 0, \alpha_{N-m}, \dots, \alpha_N) \succeq \beta$. It is clear that each such exponent is dominated by $\alpha$. Corollary 2.3 shows that each $p(\alpha; x)$ is harmonic.

Proposition 2.6. *The set* $\{p(\alpha; x) : \alpha \in \mathbb{Z}_+^N, \alpha_N = 0$ *or* $1\}$ *is an orthogonal basis for* $\mathcal{H}$*.*

Proof. By Lemma 2.4 this set has the correct number of elements of each degree. The fact that any two of these polynomials, say $p(\alpha; x)$ and $p(\beta; x)$, are orthogonal follows either from $\alpha_i, \beta_i$ having opposite parity for some $i$, or from converting the integral over the sphere to one over the simplex. Indeed suppose $f(x_1^2, \dots, x_N^2)$ is a polynomial then

$$\int_S f(x_1^2, \dots, x_N^2) h(x)^2 dm(x) = c \int_E f(t_1, \dots, t_N) \prod_{i=1}^N t_i^{k_i - 1/2} dt_1 \dots dt_{N-1},$$

where $c$ is a constant (evaluated by means of a Dirichlet-type integral), and $E$ is the $(N-1)$-dimensional simplex $\{t \in \mathbb{R}^N : \sum_{i=1}^N t_i = 1$ and each $t_i \geq 0\}$. Suppose that $f(x_1^2, \dots, x_N^2) = \prod_{i=1}^{N-1} q_i(x_i^2, \sum_{j>i} x_j^2)$ where $q_i$ is a polynomial homogeneous of degree $n_i$. Make the change of variables: $t_1 = y_1, t_i = y_i \prod_{j<i}(1 - y_j)$ for $1 < i < N$, $t_N = \prod_{j=1}^{N-1}(1 - y_j)$. Let $A_i := \sum_{j>i}(k_j + n_j) + \frac{N-i}{2} - 1$ for $1 \leq i \leq N-1$ (here $n_N = 0$); then the displayed integral equals

$$\prod_{i=1}^{N-1} \int_0^1 q_i(y_i, 1 - y_i) y_i^{k_i - 1/2} (1 - y_i)^{A_i} dy_i.$$

Suppose $\alpha, \beta \in \mathbb{Z}_+^N$ have the same parity and $\alpha \neq \beta$; let $\alpha_i = 2a_i + \epsilon_i$ and $\beta_i = 2b_i + \epsilon_i$ as in Definition 2.5, with $a_N = b_N = 0$; then $\int_S p(\alpha; x)p(\beta; x)h(x)^2 dm(x)$ is of the displayed form with $q_i(y_i, 1-y_i) = y_i^{\epsilon_i} P_{a_i}(2y_i - 1)P_{b_i}(2y_i - 1)$ (suppressing the indices for now; also $q_{N-1}$ is multiplied by $y_N^{\epsilon_N}$) and $n_i = a_i + b_i + \epsilon_i$ (add $\epsilon_N$ to $n_{N-1}$) for each $i < N$. As $\alpha \neq \beta$ there is a unique $i$ such that $a_i \neq b_i$ and $a_j = b_j$ for $j > i$. The term labeled by $i$ in the product has $A_i = \sum_{j>i}(k_j + \alpha_j) + \frac{N-i}{2} - 1$. By Definition 2.5 both the Jacobi polynomials in $q_i$ have the same indices $(A_i, k_i + \epsilon_i - \frac{1}{2})$ and this makes the integral zero.□

This basis of polynomials on the sphere was also used in Xu (1997). In Dunkl (1989, Theorem 2.1) it was shown that the adjoint of $T_i$ in $\mathcal{H}$ is a scalar multiple of $\theta x_i - |x|^2 T_i$, where $\theta := N - 4 + 2\sum_{v \in R_+} k_v + 2\sum_{i=1}^N x_i \frac{\partial}{\partial x_i}$, which is of course constant on each $\mathcal{P}_n$ (the scalar multiple is $\frac{\theta+2}{\theta}$ symbolically). Thus $S_i := \theta x_i T_i - |x|^2 T_i^2$ and $R_{ij} := (x_i T_j - x_j T_i)^2$ are self-adjoint ($1 \leq i \leq N$ and $i \neq j$ respectively). The second type comes from $(T_i^* T_j - T_j^* T_i)^2 = (\theta + 2)^2 (x_i T_j - x_j T_i)^2$.

Theorem 2.7. *The set of self-adjoint operators* $U_i := S_i + \sum_{j=1}^{i-1} R_{ij}$, $i = 1, \ldots, N$ *is pairwise commutative (for the group* $W = \mathbb{Z}_2^N$*).*

Proof. The operators $\{R_{ij}\}$ commute amongst each other whenever the indices belong to disjoint sets, also $S_i$ commutes with $R_{jm}$ if $i \notin \{j, m\}$. Thus it suffices to show that the commutators $[S_j, S_i + R_{ij}] = 0$ for $j < i$ and $[R_{ij}, R_{im} + R_{jm}] = 0$ for $i < j < m$, (where $[A_1, A_2] := A_1 A_2 - A_2 A_1$ for operators $A_1, A_2$). These commutations can be proven with the Maple techniques described above with $N = 3$ and $i = 1, j = 2, m = 3$; for this particular group the operators $T_i$ and $R_{ij}$ only act on variables with corresponding labels. Once the relation $[S_1, S_2 + R_{12}] = 0$ has been established for $N = 3$, the general case follows by replacing $x_3^2$ by $\sum_{i=3}^N x_i^2$ and $k_3$ by $N - 3 + \sum_{i=3}^N k_i$; these are constants as far as $T_1, T_2, R_{12}$ are concerned, and $\theta$ uses only the total degree, unchanged by this trick. The implementation of $\sigma_i$ and $T_i$ can be done by modifying the type A code, replacing the $j1$ loop with the line

*sm1* := *sm1* + *k*[*i*]* quo(*tm1*-subs(*x*[ *i*]=-*x*[*i*], *tm1*), *x*[*i*], *x*[*i*]) * subs(*x*[*i*]=-*x*[*i* ], *tm2*).□

In the Maple calculation the operator $\theta$ is replaced by the equivalent $N - 4 + 2\sum_{i=1}^N x_i T_i + 2\sum_{v \in R_+} k_v \sigma_v$. A Maple V Release 4 worksheet implementing these calculations is accessible on the Web, with URL `http://www.math.virginia.edu/~cfd5z/jsc-cd.mws`.

Theorem 2.8. *The operators* $U_i$ *are triangular with respect to the dominance order on monomials and* $U_i p(\alpha; x) = \kappa_i(\alpha) p(\alpha; x)$, *where* $\kappa_i(\alpha) = (\alpha_i + 2k_i\epsilon_i)(N + \alpha_i + 2k_i\epsilon_i - i - 2 + 2\sum_{j=i+1}^N (\alpha_j + k_j)) - (1 + 2(1 - 2\epsilon_i)k_i)\sum_{j=1}^{i-1}(\alpha_j + 2k_j\epsilon_j)$ *and* $\epsilon_i = 0$ *or* $1$ *when* $\alpha_i$ *is even or odd, respectively; for* $1 \leq i \leq N - 1$.

Proof. The expansion of $U_i$ yields $(\theta x_i T_i - x_i^2 T_i^2 - x_i T_i \sum_{j<i} T_j x_j - T_i x_i \sum_{j<i} x_j T_j) + x_i^2 (\sum_{j<i} T_j^2) - (\sum_{j>i} x_j^2) T_i^2$. Note that $T_j x_j^{\alpha_j} = (\alpha_j + 2k_j\epsilon_j) x_j^{\alpha_j - 1}$ for any $j$. Acting on any monomial $x^\beta$ the first part produces a multiple of $x^\beta$, the second and third parts produce linear combinations of $x^\gamma$ with $\gamma = (\ldots, \beta_j - 2, \ldots, \beta_i + 2, \ldots)$, $(j < i)$ and $\gamma = (\ldots, \beta_i - 2, \ldots, \beta_j + 2, \ldots)$, $(i < j)$ respectively; both cases satisfy $\beta \succ \gamma$. Because the operators $U_i$ commute pairwise and are triangular in the dominance order, there must be a complete set of simultaneous eigenvectors, and for each $\alpha \in \mathbb{Z}_+^N$ with $\alpha_N = 0$ or 1 there is one eigenvector with leading term $x^\alpha$. The eigenvectors must be pairwise orthogonal because they have different eigenvalues (the diagonal terms), thus the polynomials $p(\alpha; x)$ are eigenvectors (being the only orthogonal decomposition of $\mathcal{H}$ which has a triangular expansion in terms of monomials). The eigenvalue $\kappa_i(\alpha)$ is found by computing the coefficient of $x^\alpha$ in $U_i x^\alpha$.□

It is not difficult to show that $\sum_{i=1}^N U_i = (\sum_{i=1}^N x_i T_i)(\sum_{i=1}^N x_i T_i - 2)$. For the case when each $\alpha_i$ is even, the polynomials $p(\alpha; x)$ appeared in Koornwinder's survey paper (Koornwinder, 1975); the purely differential version of the operators $S_i, R_{ij}$ and their commutativity were studied in Kalnins *et al.* (1991); and a hypergroup structure was recently obtained by Koornwinder and Schwartz (1997) (with the simplex being the domain of orthogonality). The problem of Poisson kernels and Cesaro summability of expansions in $\{p(\alpha; x)\}$ for the general case (arbitrary parity of $\alpha_i$) was investigated by Xu (1997).

This paper has shown how Maple can be used to prove identities in the algebra generated by Dunkl operators, multiplication by coordinate functions and group translation, essentially in polynomial arithmetic. This technique allowed a conceptual proof of a complete orthogonal decomposition of the harmonic polynomials associated with the special case $W = \mathbb{Z}_2^N$. Future research will aim at constructing analogous decompositions for indecomposable reflection groups of rank $\geq 3$.

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